GEOMETRY QUALIFYING EXAM, FALL 2022

Instructions: Each problem is worth 20 points. Your grade will be determined by your 5 best solutions. You may submit more than 5, but are not required to do so.

(1) (a) Give an example of a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which there exists $p \in \mathbb{R}^2$ with *nonzero* Jacobian matrix $J_f(p) = \left[\frac{\partial f^i}{\partial x^j}(p)\right]$ satisfying $\det J_f(p) = 0$.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map satisfying the *Cauchy-Riemann equations*:

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2} \quad \text{and} \quad \frac{\partial f^1}{\partial x^2} = -\frac{\partial f^2}{\partial x^1}.$$

Prove that for $p \in \mathbb{R}^2$, $J_f(p) = 0$ if and only if $\det J_f(p) = 0$.

(c) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the Cauchy-Riemann equations. Prove that at points $p \in \mathbb{R}^2$ where f is a local-diffeomorphism, the local inverse of f also satisfies the Cauchy-Riemann equations.

(2) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid z - x^2 - y^2 = 1\}$.

(a) Prove M is a regular submanifold that is diffeomorphic to $\mathbb{R}^2$.

(b) Prove that $\omega = z dx \wedge dy \in \Omega^2(\mathbb{R}^3)$ is not an exact two form on $\mathbb{R}^3$.

(c) Prove the restriction $\omega_{|_M} \in \Omega^2(M)$ is an exact two form on M.

(3) Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \chi(\mathbb{R}^2)$.

(a) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = e^{x^2+y^2}$. Prove if $g : \mathbb{R}^2 \to \mathbb{R}$ is smooth, then

$$X(g) = 0 \iff X(fg) = 0.$$

(b) Suppose $Y \in \chi(\mathbb{R}^2)$ is a smooth vector field having the same restriction as X to the unit circle. Show there exists $p \in \mathbb{R}^2$ such that $Y(p) = 0$.

(4) Let $V, W \in \chi(\mathbb{R}^3)$ be smooth vector fields defined by

$$V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

$$W = \frac{\partial}{\partial y}.$$

(a) Determine the smooth functions $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i \in \{1, 2, 3\}$, defined by

$$[V, W] = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z}.$$

(b) Show for a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, $V(f) = W(f) = 0$ if and only if f is a constant function.

(5) For each $\theta \in \mathbb{R}$ and $\lambda \in (0, \infty)$ define diffeomorphisms

$$R_\theta, D_\lambda : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

by

$$R_\theta(x, y) = (\cos(\theta)x + \sin(\theta)y, -\sin(\theta)x + \cos(\theta)y)$$

and

$$D_\lambda(x, y) = (\lambda x, \lambda y).$$

Determine the smooth functions $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that the smooth 2-form $\omega = f dx \wedge dy$ has the property that for every $(\theta, \lambda) \in \mathbb{R} \times (0, \infty)$, $R_\theta^*(\omega) = \omega$ and $D_\lambda^*(\omega) = \omega$.

(6) Let $X \in \chi(\mathbb{R}^2)$ be the smooth vector field defined by

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the smooth function defined by $f(x, y) = xy$. Show $X(f) = 0$.

(b) Suppose that $c(t)$ is an integral curve. Explain why part (a) implies $f(c(t))$ is constant as a function of t.

(c) Let $p_0 = (x_0, y_0) \in \mathbb{R}^2$. Find the maximal integral curve $c(t)$ of X with $c(0) = p_0$. Verify that $f(c(t))$ is constant in t directly using your formula for the integral curve $c(t)$.

(7) Let $\lambda \in \mathbb{R}$ and let $F_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (xe^y - \lambda y, xe^y + y)$.

(a) Determine those $\lambda \in \mathbb{R}$ for which F_λ is a diffeomorphism of $\mathbb{R}^2$.

(b) Determine those $\lambda \in \mathbb{R}$ for which F_λ is an *orientation preserving* diffeomorphism of $\mathbb{R}^2$.

(8) Let M be a smooth manifold. Let $\omega \in \Omega^1(M)$ and let $f : M \to \mathbb{R}$ be a smooth and positive function. Show that if $d(f\omega) = 0$, then $\omega \wedge d\omega = 0$.